Simplexes and Chains

Affine Simplexes:
Suppose X and Y are vector spaces (over the same field). We say $f: X \to Y$ is an affine map if

$f - f(0)$ is linear,
i.e. $f(x) = f(0) + Ax$, for some $A \in L(X, Y)$.
For example, if $f: \mathbb{R}^n \to \mathbb{R}^m$, $f(x) = A \cdot x + b$, $b \in \mathbb{R}^m$ being fixed and A is an $m \times n$ matrix, then f defines an affine map. Note that for such maps to be described fully, we need only to know $f(0)$ and $f(e_i)$, $1 \le i \le n$.

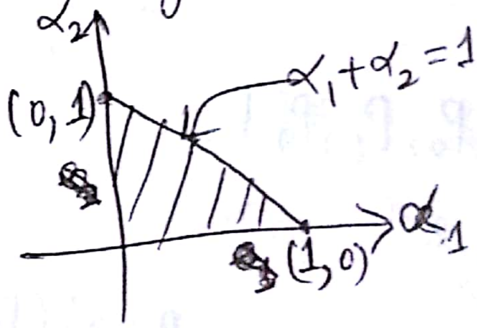
Standard Simplex Q^k: Recall that the standard simplex Q^k was defined as:

$$Q^k = \left\{ u \in \mathbb{R}^k : u = \sum_{j=1}^{k} \alpha_j e_j ,\ \alpha_j \ge 0 \ \forall\ 1 \le j \le k, \text{ and } \sum_{j=1}^{k} \alpha_j \le 1 \right\}.$$

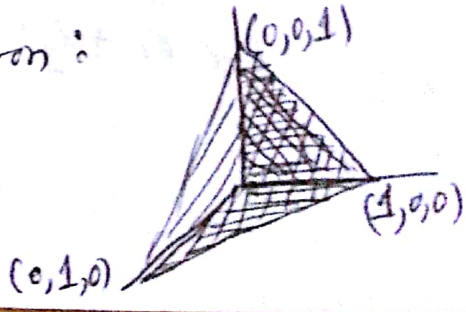
In particular,

$Q^1 = [0,1] \subset \mathbb{R}$

Q^2 is the triangular region:



Q^3 is the tetrahedron:



Now we define

<u>Oriented affine k-Simplex</u>: Suppose $P_0, P_1, \cdots, P_k$ are points in $\mathbb{R}^n$ $(k \geq 1)$. We define the oriented affine simplex

$$\sigma = [P_0, P_1, \cdots, P_k] \quad \text{———} (1)$$

to be the k-~~simplex~~ surface in $\mathbb{R}^n$ with parameter domain Q^k given by the affine mapping

$$\sigma\left(\sum_{j=1}^{k} \alpha_j e_j\right) = P_0 + \sum_{j=1}^{k} \alpha_j (P_j - P_0), \quad \text{———} (2)$$

or, equivalently,

$$\sigma(0) = P_0, \quad \sigma(e_i) = P_i - P_0, \quad 1 \leq i \leq k.$$

~~Special~~ If $k = 1$, $\sigma = [P_0, P_1]$, i.e.

$$\sigma : [0, 1] \to \mathbb{R}^n,$$

$$\sigma(t) = P_0 + t(P_1 - P_0)$$
$$= (1-t)P_0 + tP_1,$$

which is the directed line segment from P_0 to P_1.

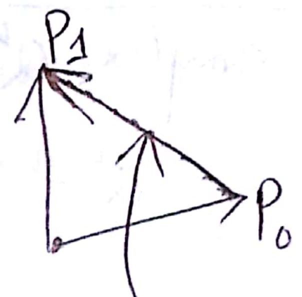
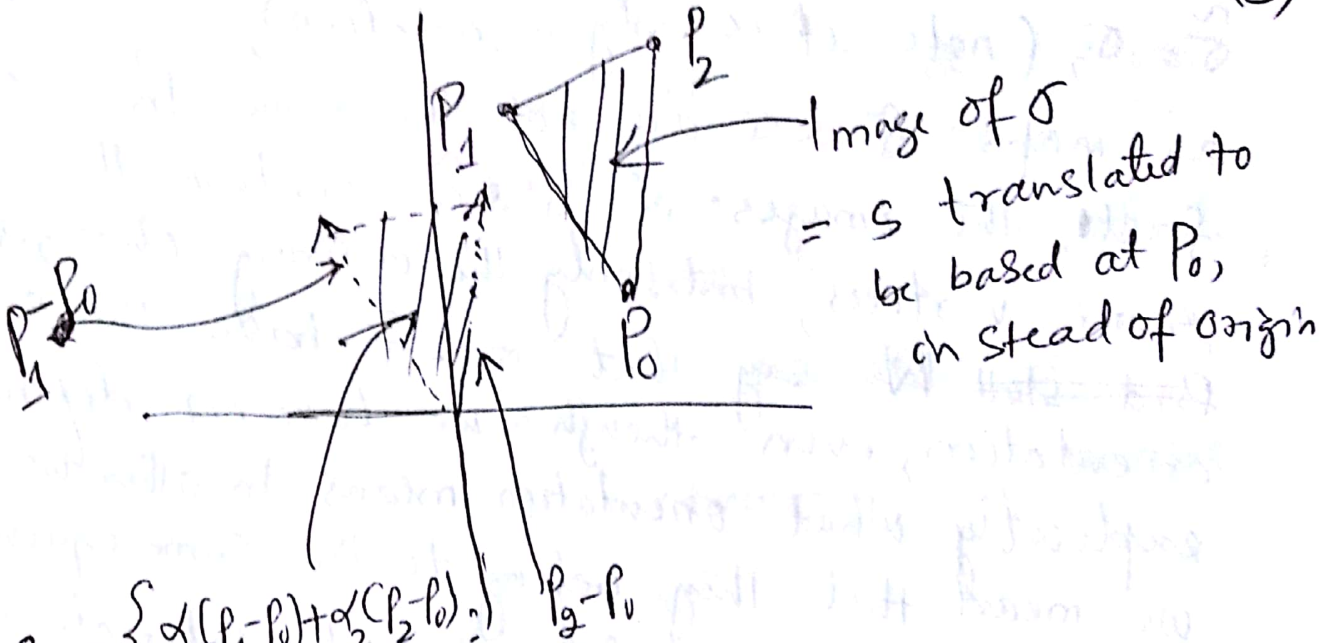


Image of σ

$k = 2$, $\sigma = [P_0, P_1, P_2]$

i.e. $\sigma : Q^2 \to \mathbb{R}^n$

$$\sigma(\alpha_1 e_1 + \alpha_2 e_2) = P_0 + \alpha_1 (P_1 - P_0) + \alpha_2 (P_2 - P_0)$$



$$S = \{\alpha_1(p_1 - p_0) + \alpha_2(p_2 - p_0) : \alpha_1, \alpha_2 \geq 0, \alpha_1 + \alpha_2 \leq 1\}$$

and so on.

The word 'oriented' is used to emphasize that in the definition (1), the ordering of the vectors $p_0, p_1, \dots, p_k$ is taken into account. Suppose, for example, $\{i_0, i_1, \dots, i_k\}$ is a permutation of the ordered set $\{0, 1, \dots, k\}$, and

$$\tilde{\sigma} = [p_{i_0}, p_{i_1}, \dots, p_{i_k}],$$

then we adopt the notation

$$\tilde{\sigma} = \varepsilon(i_0, i_1, \dots, i_k)\sigma,$$

where, as was used earlier,

$$\varepsilon(i_1, \dots, i_k) = \prod_{p<q} \operatorname{sgn}(i_q - i_p).$$

In ~~order~~ other words, $\tilde{\sigma} = \pm\sigma$. But a precaution needs to be taken that, when $\varepsilon = 1$, i.e.

$\tilde{\sigma} = \sigma$, (note it is only a notation), as maps $\tilde{\sigma}$ & σ are not the same. In fact, the images of σ & $\tilde{\sigma}$ contain the same vertices, but only the ordering changes. ~~But still~~ We say that σ & $\tilde{\sigma}$ have the same orientation, even though we have not defined explicitly what orientation means. In other words, we mean that they belong to the same equivalence class. We shall see later that the integration of a k-form does not change if we replace σ by $\tilde{\sigma}$ (when they have the same orientation).

If $\varepsilon = 1$, we say σ & $\tilde{\sigma}$ have the <u>same orientation</u>, and if $\varepsilon = -1$, σ and $\tilde{\sigma}$ have opposite orientation.

<u>Exercise</u> (1) Suppose $n=4, k=2$,

$$\sigma = [P_0, P_1, P_2], \quad \sigma_1 = [P_0, P_2, P_1], \quad \sigma_2 = [P_1, P_2, P_0],$$
$$\sigma_3 = [P_2, P_0, P_1], \quad \sigma_4 = [P_1, P_0, P_2].$$

Decide which of the σ_i's have orientation same as σ, and which have opposite.

(2) $n=4$, $k=3$,

$$\sigma = [P_0, P_1, P_2, P_3], \quad \sigma_1 = [P_0, P_2, P_3, P_1], \quad \sigma_2 = [P_1, P_0, P_3, P_2],$$
$$\sigma_3 = [P_2, P_1, P_0, P_3], \quad \sigma_4 = [P_0, P_3, P_1, P_2]$$
$$\sigma_5 = [P_1, P_3, P_2, P_0]$$

Decide which of the σ_i's have same orientation as that of σ, and which have opposite orientation to that of σ. $\square$

If $n=k$, and the vectors $P_j - P_0$, $1 \leq j \leq k$, are linearly independent, and we write the affine transformation ~~σ as~~

$\sigma = [P_0, P_1, \ldots, P_k]$ as

$$P_0 + Au, \qquad (3)$$

where $A \in L(\mathbb{R}^k, \mathbb{R}^n)$

$$Ae_j = P_j - P_0,$$

then A is invertible. So $\det A$, which is determinant of the Jacobian of σ, is non zero.

Thus, in this case, we can define orientation explicitly as:

σ is positively oriented if $\det A > 0$

and is negatively oriented if $\det A < 0$.

Example: $k=n=4$,

$\sigma = [P_0, P_1, P_2, P_3, P_4]$

$$P_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}.$$

Determine whether σ is positively oriented or negatively.

So far we have assumed $k \geq 1$. An oriented 0-simplex (in $\mathbb{R}^n$) is defined to be a point (in $\mathbb{R}^n$) with a sign attached. For example $\sigma = \pm P_0$ is an ~~0-sim~~ oriented 0-simplex in an open set $E \subset \mathbb{R}^n$ if $P_0 \in E$ (and even if $-P_0 \notin E$).

Integration: If $\sigma = \varepsilon P_0$, ~~where~~ in $E \subset \mathbb{R}^n$, where $\varepsilon = \pm 1$, and if f is a 0-form in E (i.e. a real-valued function on E), we define

$$\int_\sigma f = \varepsilon f(P_0) \quad \text{---} (4)$$

If σ is an oriented affine k-simplex in E, $k \geq 1$, then since it is a k-surface with parameter domain Q^k, any k-form in E can be integrated over σ in the usual way.

The next result shows that the integration of a k-form over two oriented affine k-simplexes having the same orientation gives the same value.

Thm 1: Let σ be an oriented affine k-simplex in an open set $E \subset \mathbb{R}^n$. If $\tilde{\sigma} = \varepsilon \sigma$, $(\varepsilon = \pm 1)$, then for any k-form ω in E,

$$\int_{\tilde{\sigma}} \omega = \varepsilon \int_\sigma \omega \quad \text{---} (5).$$

<u>Proof:</u> If $k=0$, then (5) follows from the definition (4). So assume $k\geq 1$ and $\sigma=[P_0,P_1,\cdots,P_k]$. In view of additivity of the integral, enough to consider ω to be a basic k-form, say, $\omega=a(x)dx_I$, $I=\{i_1,i_2,\cdots,i_k\}$.

<u>Case 1:</u> $1\leq j\leq k$ and $\tilde{\sigma}$ obtained from σ by interchanging P_0 and P_j, i.e. $\tilde{\sigma}=[P_j,P_1,\cdots,P_{j-1},P_0,P_{j+1},\cdots,P_k]$

Then $\tilde{\sigma}=\varepsilon\sigma$, where

$$\varepsilon=(-1)^{2j-1}=-1$$

(Note how many indices to the right of P_j are less than j, and so on. Thus $j+1\cdot(j-1)=2j-1$)

Now $\tilde{\sigma}(0)=P_j$

$\tilde{\sigma}(e_i)=P_i$, $1\leq i\leq j-1$, $j+1\leq i\leq k$

$\tilde{\sigma}(e_j)=P_0$.

Thus $\tilde{\sigma}\left(\sum_{i\geq 1}^{k}\alpha_i e_i\right)=P_j+\sum_{\substack{i=1\\ i\neq j}}^{k}\alpha_i(P_i-P_j)+\alpha_j(P_0-P_j)$

So we may write $\tilde{\sigma}$ as

$$\tilde{\sigma}(u)=P_j+Bu,$$

where $B\in L(\mathbb{R}^k,\mathbb{R}^n)$ given by

$$Be_i=P_i-P_j,\quad 1\leq i\leq j-1,\ j+1\leq i\leq k$$
$$=P_0-P_j,\ \text{if } i=j.$$

On the other hand, we can write σ as

$$\sigma(u)=P_0+Au,$$

where $A\in L(\mathbb{R}^k,\mathbb{R}^n)$ given by

$A e_i = P_i - P_0$, $1 \leq i \leq k$,
$= x_i$, say.

Then $B e_i = x_i - x_j$, $\forall\, i \neq j$
$= -x_j$, $i = j$.

So, w.r.t the standard bases on $\mathbb{R}^n$, the matrices $\tilde{A}$ of A and $\tilde{B}$ of B are given by

$$\tilde{A} = [x_1 \;\; x_2 \;\; \cdots \;\; x_k]_{n\times k}$$

$$\tilde{B} = [(x_1 - x_j)\;(x_2 - x_j) \cdots (x_{j-1} - x_j),\;(-x_j)\;(x_{j+1} - x_j) \cdots (x_k - x_j)]_{n\times k}$$

Now,
$$\int_\sigma w = \int_{Q^k} a(\sigma(u)) \det J(u)\, du$$

and
$$\int_{\tilde{\sigma}} w = \int_{Q^k} a(\tilde{\sigma}(u)) \det \tilde{J}(u)\, du, \quad \text{---}(6)$$

where $J(u) = \dfrac{\partial(\sigma_{i_1}, \ldots, \sigma_{i_k})}{\partial(u_1, \ldots, u_k)}$ is the matrix obtained from the Jacobian matrix $J_\sigma = \tilde{A}$ of σ by taking the $i_1^{th}, i_2^{th}, \ldots, i_k^{th}$ rows, and similarly

$\tilde{J}(u) = \dfrac{\partial(\tilde{\sigma}_{i_1}, \ldots, \tilde{\sigma}_{i_k})}{\partial(u_1, \ldots, u_k)}$ is the matrix obtained from the Jacobian matrix $J_{\tilde{\sigma}} = \tilde{B}$ of $\tilde{\sigma}$ by

Considering $i_1^{th}, i_2^{th}, \dots, i_k^{th}$ rows. So if we replace $\tilde{B}$ by B_1, where ~~B_1 is ob~~

$$B_1 = [x_1 \; x_2 \; \cdots \; x_{j-1} \; -x_j \; x_{j+1} \; \cdots \; x_k]$$

is the matrix obtained from $\tilde{B}$ by subtracting the jth column from every other column, then the determinant in (6) remains unchanged. Therefore

$$\det \tilde{J}(u) = -\det J(u)$$

Thus substituting in (6),

$$\int_{\tilde{\sigma}} w = -\det J(u) \int_{Q^k} a(\tilde{\sigma}(u))\, du \quad \text{--- (7)}$$

(Note that ~~det~~ $J(u)$ is a constant matrix since σ is an affine map)

Claim: $$\int_{Q^k} a(\tilde{\sigma}(u))\, du = \int_{Q^k} a(\sigma(u))\, du \quad \text{--- (8)}$$

(Once we prove (8), we get from (7)

$$\int_{\tilde{\sigma}} w = -\int_{\sigma} w = \varepsilon \int_{\sigma} w \quad (\text{here } \varepsilon = -1)$$

Proof of claim: We achieve this by producing an affine map $W : Q^k \to Q^k$ such that

$$\tilde{\sigma}(u) = \sigma(W(u)), \quad |\det J_W| = |-1| = 1$$

To get W, notice that

$$\sigma(e_j) = P_j = \tilde{\sigma}(0).$$

So take $W(0) = e_j$

Illy, $W(e_j) = 0$ & $W(e_i) = e_i$, $i \neq j$.

Thus, for $u = \sum_{i\geq 1}^{k} \alpha_i e_i \in Q^k$,

$$W(u) = L\cdot u + e_j,$$

where $L = \begin{bmatrix}(e_1 - e_j) & (e_2 - e_j) & \cdots & (e_{j-1} - e_j) & (-e_j), & (e_{j+1} - e_j) & \cdots & e_k - e_j\end{bmatrix}_{k\times k}.$

So as in the proof above, subtracting jth column from every other column, we get the matrix

$$L' = [\, e_1 \cdots e_{j-1} \; -e_j \; e_{j+1} \cdots e_k \,].$$

Thus $\det J_W = -1$

Now we can use the change of variable theorem: (notice that $W(Q^k) = Q^k$)

$$\int_{Q^k} a(\tilde{\sigma}(u))\, du = \int_{Q^k} a(\sigma(W(u)))\, du$$

$$= \int_{Q^k} a(\sigma(W(u)))\, |\det J_W(u)|\, du$$

$$= \int_{Q^k} a(\sigma(v))\, dv,$$

which proves the claim.

<u>Case 2</u>: Suppose $0 < i < j \le k$ and $\tilde{\sigma}$ is obtained from σ by interchanging P_i and P_j;

i.e. $\tilde{\sigma} = [P_0, P_1, \ldots, P_{i-1}, P_j, P_{i+1}, \ldots, P_{j-1}, P_i, P_{j+1}, \ldots, P_k]$.

Then $\tilde{\sigma}(u) = P_0 + C \cdot u$,

where the matrix C has the same columns as $\tilde{A}$ except that the ith and jth column have been interchanged. In this case,

$\tilde{\sigma} = \varepsilon \sigma$, where

$\varepsilon = (-1)^{2(j-i)-1} = -1$ which are less than j =

(No. of indices to the right of P_j
$j-1-i+1 = j-i$
& one each to the right of $P_{i+1}, \ldots, P_{j-1}$)

Also as in Case 1, $\det \tilde{J}(u) = -\det J(u)$. Moreover, (8) holds in this case as well, In fact, we can choose

$$Wu = L \cdot u + 0, \quad L = [e_1, \ldots, e_{i-1}, e_j, e_{i+1}, \ldots, e_{j-1}, e_i, e_{j+1}, \ldots e_k]_{k \times k}$$

& $\det L = -1$. Therefore, we get

$$\int_{\tilde{\sigma}} \omega = -\det J(u) \int_{Q^k} a(\tilde{\sigma}(u))$$

$$= -\det J(u) \int_{Q^k} a(\sigma(W(u))) \, |\det J_W(u)| \, du$$

$$= -\det J(u) \int_{Q^k} a(\sigma(u)) \, du$$

$$= - \int_\sigma \omega .$$

Case 3 (General case)

Every permutation $\{i_0', i_1', \dots, i_k'\}$ of $\{0, 1, \dots, k\}$ is a composition of the transpositions we have already ~~used~~ dealt with. Consequently, if $\tilde{\sigma} = [P_{i_0'}, P_{i_1'}, \dots, P_{i_k'}]$, then $\tilde{\sigma} = (-1)^r \sigma$, where r is the no. of transpositions that are used. Consequently, the determinant also changes by $(-1)^r$, and so the result follows from the earlier cases. This completes the proof of theorem. □

Affine chains

An affine k-chain Γ in an open set $E \subset \mathbb{R}^n$ is a collection of finitely many oriented affine k-simplexes $\sigma_1, \sigma_2, \dots, \sigma_r$ in E. These simplexes need not be distinct, a simplex may occur in Γ with finite multiplicity.

Let Γ be an affine k-chain as above.

For a k-form ω on E, we define

$$\int_\Gamma \omega = \sum_{i=1}^{r} \int_{\sigma_i} \omega . \quad \text{———} (1)$$

Remark about notation: If Φ is a k-surface in E, we have seen that

$$\Phi : D \subset \mathbb{R}^k \to E .$$

We can also view it as a map

$$\Phi : \Omega^k(E) \to \mathbb{R}$$

Where, for $\omega \in \Omega^k(E)$, $\Phi(\omega) = \int_\Phi \omega$.

Since real-valued functions can be added, we can interpret the RHS of (1) as

$$\sigma_1(\omega) + \sigma_2(\omega) + \cdots + \sigma_r(\omega)$$

i.e. $\Gamma(\omega) = \sum_{j=1}^{r} \sigma_j(\omega)$

or $\Gamma = \sigma_1 + \sigma_2 + \cdots + \sigma_r = \sum_{j=1}^{r} \sigma_j$.

But a special care needs to be taken while using this notation. For example, an oriented affine k-simplex σ is now interpreted in two different ways:

$$\sigma : Q^k \to \cancel{\mathbb{R}} E$$

$$\sigma : \Omega^k(E) \to \mathbb{R}.$$

If $\sigma_2 = -\sigma_1$ (as in the earlier notation) i.e. σ_1 & σ_2 have the same vertices but opposite orientation, and if $\Gamma = \sigma_1 + \sigma_2$, then for $\omega \in \Omega^k(E)$,

$$\Gamma(\omega) = (\sigma_1 + \sigma_2)\cdot(\omega) = \int_{\sigma_1}\omega + \int_{\sigma_2}\omega = \int_{\sigma_1}\omega - \int_{\sigma_1}\omega = 0.$$

This means $\Gamma = 0$.

But as a map $\Gamma : Q^k \to \mathbb{R}^n$, this is NOT the zero map. ~~For~~

For example,

$$\sigma_1 = [P_1, P_2], \quad \sigma_2 = [P_2, P_1]$$

Then $\sigma_2 = -\sigma_1$. But $(\sigma_1+\sigma_2)(0) = P_1 + P_2$.

Or, we may take

$$\sigma_1 = [P_0, P_1, P_2], \quad \sigma_2 = [P_1, P_0, P_2].$$

Then $\sigma_2 = -\sigma_1$. But $(\sigma_1+\sigma_2)(0) = P_0 + P_1$, and so on.

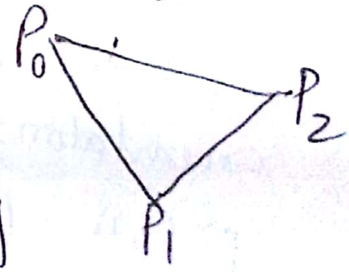
Boundaries

Let $k \geq 1$ and $\sigma = [P_0, P_1, \cdots, P_k]$ be an oriented affine k-simplex in $\mathbb{R}^n$. Define <u>the boundary $\partial\sigma$ of σ</u> to be the affine ~~k-d~~ $(k-1)$-chain

$$\partial\sigma = \sum_{i=0}^{k} (-1)^i [P_0, P_1, \cdots, P_{i-1}, P_{i+1}, \cdots, P_k] \quad \text{—— (2)}$$

<u>Examples:</u> <u>$k=1$.</u> $\sigma = [P_0, P_1]$

$$\partial\sigma = P_1 - P_0$$



<u>$k=2$</u>, $\sigma = [P_0, P_1, P_2]$

$$\partial\sigma = [P_1, P_2] - [P_0, P_2] + [P_0, P_1]$$
$$= [P_1, P_2] + [P_2, P_0] + [P_0, P_1].$$

<u>Exercise:</u> What is the boundary of $\sigma = [P_1, P_2, P_0]$?

Boundary of an affine k-chain $\Gamma = \sum_{i=1}^{r} \sigma^{(i)}$ can be defined by taking the boundaries of each simplex $\sigma^{(i)}$. That is

$$\partial \Gamma = \sum_{i=1}^{r} \partial \sigma^{(i)}.$$

Coming back to (2), for $1 \le j \le k$,

$\sigma_j = [P_0, P_1, \dots, P_{j-1}, P_{j+1}, P_{\dots}, P_k]$ is an oriented affine $(k-1)$-simplex with parameter domain Q^{k-1}, defined by

$$\sigma_j\left(\sum_{i=1}^{k-1} \alpha_i e_i\right) = P_0 + \alpha_1 (P_1 - P_0) + \dots + \alpha_{j-1}(P_{j-1} - P_0) + \alpha_j (P_{j+1} - P_0) + \dots + \alpha_{k-1}(P_k - P_0).$$

Thus

$$\sigma_j(e_i) = P_i, \text{ for } 1 \le i \le j-1$$
$$= P_{i+1}, \text{ for } j \le i \le k-1.$$

Also we can write $\sigma_j(u) = P_0 + Tu$, where $T \in L(\mathbb{R}^{k-1}, \mathbb{R}^n)$,

$$T e_i = P_i - P_0, \text{ for } 1 \le i \le j-1$$
$$= P_{i+1} - P_0, \text{ for } j \le i \le k-1.$$

And $\sigma_0 = [P_1, \dots, P_k]$ is an oriented affine $(k-1)$-simplex defined on Q^{k-1} by

$$\sigma_0\left(\sum_{i=1}^{k-1} \alpha_i e_i\right) = P_1 + \sum_{i=1}^{k-1} \alpha_i (P_{i+1} - P_1).$$

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This means,

$$\sigma_0(e_i) = P_{i+1}, \quad 1 \le i \le k-1.$$

We can also write as

$$\sigma_0(u) = P_1 + Su,$$

where $S \in L(\mathbb{R}^{k-1}, \mathbb{R}^n)$

$$S e_i = P_{i+1} - P_1, \quad 1 \le i \le k-1.$$